University *of* Virginia

Department *of* Mathematics

On the Distributions of Point Counts on Hypergeometric Varieties

This thesis is on arithmetic statistics that arise from arithmetic geometry. In particular, I prove arithmetic-statistical results about the distributions of points on hypergeometric varieties. As modular forms are central objects in these results, I provide a new proof of the celebrated Eichler–Selberg trace formula for levels dividing 4. The central elements of this paper are elliptic curves, finite field hypergeometric functions, harmonic Maass forms, and holomorphic modular forms.

It is natural to study statistical questions about the number of finite-field points on algebraic varieties. One of the most famous such questions is the Sato–Tate conjecture on the distribution of the traces of Frobenius for a fixed non-CM elliptic curves as one goes over all primes of good reduction. In their landmark work, Richard Taylor and his collaborators proved this conjecture using deep tools from the analytic theory of automorphic forms, l-adic Galois representations and étale cohomology. Inspired by this deep conjecture, I determine the limiting distribution of Frobenius traces for the family of Legendre elliptic curves and a special family of $K3$ surfaces. In order to prove these results, in joint work with Saikia and Ono, I use results from arithmetic geometry, the theory of holomorphic modular forms and harmonic Maass forms, and the method of moments.

In light of the fact that the limiting distribution for the $K3$ surfaces has vertical asymptotics, I explicitly bound the error in the limiting distribution for $K3$ surfaces. In order to do so, I use Rankin–Selberg unfolding, the theory of newforms, and the Beurling–Selberg polynomials.

The Eichler–Selberg trace formulas express the traces of Hecke operators on a spaces of cusp forms in terms of weighted sums of Hurwitz–Kronecker class numbers. For cusp forms on $\mathrm{SL}_2(\mathbb{Z})$, Zagier proved these formulas by cleverly making use of the weight $3/2$ nonholomorphic Eisenstein series he discovered in the 1970s. Using Zagier's method, in joint work with Ono, I prove the Eichler–Selberg trace formulas for $\Gamma_0(2)$ and $\Gamma_0(4)$. To do this, I use Zagier's Eisenstein series, Rankin–Selberg unfolding, the Petersson inner product, and the theory of holomorphic modular forms.

In their famous work, Feit and Fine count the number of pairs of commuting $n \times n$ matrices with entries in a finite field. This can be framed as counting $\mathbb{F}_p$-points on the commuting variety defined by (A, B) of $n \times n$ matrices which satisfy the equation $AB - BA = 0_n$. Motivated by this, in joint work with Huang and Ono, I count the number of $n \times n$ matrix points on Legendre elliptic curves and $K3$ surfaces in terms of finite field hypergeometric functions and partitions. Using this explicit point count, I determine the limiting distribution of the "random part" of the matrix point counts as the finite field grows. In order to do this, I use results of Huang that connect the number of matrix points to the zeta function of a variety.

Hasan Saad

Advisor Ken Ono

Defense Date 3/25/2024 @ 1 P.M.

Defense Location Kerchof Hall 111

Homepage hasansaad2.github.io